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Bounds for the fractal dimension of space

Andreas Schäfer† and Berndt Müller‡

† Gesellschaft für Schwerionenforschung, Postfach 110541, D-6100 Darmstadt, West Germany

‡ Institut für Theoretische Physik, J W Goethe-Universität, Postfach 111932, D-6000 Frankfurt, West Germany

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Abstract. We discuss the possibility that space has to be described as being of non-integer dimension. We elaborate on some of the problems associated with this idea and investigate the chance to observe effects due to such a fractional dimension. Two experiments allow us to derive rather stringent bounds for the deviation of the space dimension from the value three. These are the perihelion shift of planetary motion and the Lamb shift in hydrogen.

1. Introduction

The question we deal with in this paper can be formulated as follows. How do we know that the dimension of space D is really three and not, e.g., 3.000 0001? The most natural answer to this question, namely that non-integer dimensions are inconceivable, is not valid. Non-integer dimension is a well defined mathematical concept which may be, and, in fact, has been applied to the description of physical phenomena. It shares with many concepts of modern physics the property of being accessible to intuitive understanding only with considerable effort.

The question of whether spacetime has to be described as being endowed with integer or fractional dimension is a basic problem of conceptual significance. Even the mere fact that space can be thought of as being of fractional dimension could lead to a deeper understanding, as it demands a physical argument as to why an integer dimension should be favoured.

We want to discuss here two rather different examples of the physical significance which can be attributed to the appearance of 'fractional dimensions'.

(1) Based on the original suggestion of Kaluza and Klein [1, 2] many models have been developed in the past few years which assume the existence of additional spacetime dimensions [3, 4], e.g. a total of eleven dimensions. These additional dimensions are supposed to be 'compactified', which means that one assumes them to be rolled up with such a small curvature radius that any excitation in these additional degrees of freedom has an energy of the order of 10^{16} GeV. With these assumptions it is obvious that excitations in the compactified dimensions cannot significantly influence the processes at energies attainable today. However it might be possible to detect relics of the presence of these underlying dimensions. As an example, let us discuss the situation for a harmonic oscillator in D dimensions of which all but three are compactified. For an oscillator length l_0 much larger than the curvature radius ρ of the compactified dimensions, its low energy spectrum will be that of a three-dimensional

oscillator, whereas for $l_0 \ll \rho$ the influence of the curvature is negligible and we get, e.g. for a model with ten space dimensions, the spectrum of a ten-dimensional harmonic oscillator. Obviously the transition between the two extremes will take place smoothly. It is then possible to attribute an effective dimension to this spectrum. The precise definition of d_{eff} is not of importance for our general argument. Nevertheless we want to give an illustrative example. Let us assume that the compactification is done by closing every auxiliary dimension separately, i.e. that the structure of space is given by $R^3 \times S^1 \times S^1 \times \ldots \times S^1$. Then all the space coordinates decouple. With respect to any space dimension, one finds a ground state and a first excited state. Let us now denote by E_i, $i = 1, 2, \ldots, 10$, the energy of the phonon corresponding to an excitation in the ith space dimension. Let us furthermore define $E_{\min}$ as the minimum value of the E_i. With these definitions one can define d_{eff} as

$$d_{\text{eff}}(l_0) = \sum_i E_{\min}/E_i. \tag{1}$$

For $l_0 \ll \rho$ all space dimensions are equivalent and therefore all the E_i are equal. Thus d_{eff} becomes the degeneracy of the first excited level in the free ten-dimensional harmonic oscillator, which is ten. On the other hand, for $l_0 \gg \rho$ we have $E_{\min} = E_1 = E_2 = E_3 \ll E_i$, $i = 4, 5, \ldots, 10$ and consequently we find $d_{\text{eff}} \simeq 3$. In general we find

(i) d_{eff} depends on the energy scale, e.g. on the value of l_0, and

(ii) d_{eff} can take arbitrary non-integer values.

For our example of a harmonic oscillator, for $l_0 \gg \rho$ the effective dimensions would then be of the form

$$d_{\text{eff}} = 3 + \text{constant} \times (\rho/\lambda_c)(\rho/l_0) \tag{2}$$

where λ_c is the Compton wavelength of the particle considered.

If one wants to follow these ideas one would thus interpret D as an effective parameter and its deviation from three as a consequence of the microscopic higher-dimensional structure of space.

(2) The success of the lattice calculations [5], especially in connection with lattice gauge theories [6], shows that it is possible to interpret the four-dimensional spacetime continuum as the low energy appearance of a discrete lattice with a lattice constant which is small compared with distances relevant for physical processes. There do, however, exist lattices to which one has to attribute fractional space dimensions [7]. Therefore it is by no means clear that space, in the main, is a metric space of exactly three dimensions or whether its dimensionality differs, however slightly, from this integer value. This idea received an enormous impetus from intensive recent work [8], e.g. indicating a close relationship between these fractional dimensions and those appearing in the theoretical treatment of phase transitions. We do not want to discuss these results here. Instead let us just note the following properties of these lattices.

(i) The dimension of a 'fractal' point set can take arbitrary non-integer values.

(ii) If one introduces a finite resolution, e.g. a typical scale l_0, into the generalised definition of the dimension of a point set, one finds that the space dimension D will in general be a function of this resolution.

Within the framework of this interpretation a non-vanishing value of $D = 3$ would indicate a non-trivial microscopic lattice structure of space.

There are still more ideas [9] which can be used to motivate the assumption that space might have fractional dimension. However, as they are less well established and in any case lead to the same qualitative properties as the two interpretations we have presented, we will not discuss them.

In our contribution we want to elaborate on the arguments of [10], leading to rather stringent limits for the deviation of the D from three for two vastly different length scales [11]. This programme was also motivated by the work of Zeilinger and Svozil [12] who noted that the current discrepancy between theoretical and experimental values of the anomalous magnetic moment of the electron could be resolved if the dimensionality of space D is

$$D = 3 - (5.3 \pm 2.5) \times 10^{-7}. \tag{3}$$

The bounds we derive are more stringent and thus would rule out this interpretation unless the fractal dimension were entirely timelike. The detailed calculations will be presented in §§ 3 and 4, while some conceptual ideas and problems are discussed in § 2.

2. Some basic ideas

In our calculations we use the idea of dimensional continuation, well known from the method of dimensional regularisation [13]. There one observes that any n-point function becomes finite for suitably chosen dimensions of spacetime $d = 1 + D$. For example, for the vacuum polarisation tensor of quantum electrodynamics we obtain

$$\Pi_{\mu\nu}(k_\mu, d) = -\mathrm{i}(e^2/2\pi^4) \int_0^1 \mathrm{d}\alpha\, \pi^{d/2} \alpha(1-\alpha) M^{4-d} / [-m^2 + \alpha(1-\alpha)k^2]$$
$$\times \Gamma(2 - d/2)(g_{\mu\nu}k^2 - k_\mu k_\nu) \tag{4}$$

which is finite for $D = 0$, 1 and 2. This expression is then continued to arbitrary dimensions, and even to non-integer dimensions, just by keeping equation (4) and allowing d to be any real number. With $d = 4 - \varepsilon$ we obtain, in this specific example,

$$\Pi_{\mu\nu}(k) = -e^2/(2\pi^2)\left(\tfrac{1}{6} - \varepsilon \int_0^1 \mathrm{d}\alpha\, \alpha(1-\alpha) \log\{\pi[-m^2 + \alpha(1-\alpha)k^2]/M^2\}\right)$$
$$\times (1/\varepsilon - \gamma)(g_{\mu\nu}k^2 - k_\mu k_\nu) + \mathrm{O}(\varepsilon). \tag{5}$$

We proceed in exactly the same way. We calculate the interesting quantities as functions of the space dimension D. These expressions are then continued to arbitrary values of D by keeping the analytic expression and allowing D to become non-integer.

Although this procedure is generally accepted we want to note that it is not clear how it can be formulated in a strict mathematical manner. Obviously the continuation of equation (1) is not unambiguous. One could, e.g., add a term of the form

$$\Gamma(2 - d/2) k_\mu k_\nu f(4 - d, k^2) \sin[\pi(4 - d)] \equiv 0 \qquad \text{for } d = 1, 2, 3 \tag{6}$$

with an arbitrary function $f(\varepsilon, k^2)$. This term is zero for all values of d for which the expression (4) is well defined. Such a term would lead to an additional contribution of the form

$$(1/\varepsilon - \gamma + \mathrm{O}(\varepsilon)) k_\mu k_\nu f(\varepsilon, k^2) \pi\varepsilon = k_\mu k_\nu f(0, k^2) \tag{7}$$

which would arbitrarily change the renormalised vacuum polarisation tensor and even violate gauge invariance. One could try to avoid these problems by imposing suitable restrictions on $\Pi_{\mu\nu}(k, d)$ as a function of d but it is unclear how these restrictions can be motivated physically. As we have not found any formulation in the literature which avoids these ambiguities we are left with the feeling that the concept of dimensional continuation has been shown to lead to correct results but still waits for a mathematically rigorous definition. If these mathematical problems were overcome one could possibly even avoid the appearance of divergences in quantum field theory by assuming $\varepsilon \neq 0$. This hope is based on the fact that, using the dimensional regularisation scheme, the critical contributions are isolated in terms of the form constant ε which are finite for non-zero ε.

For our calculations we assume in the following, in line with most other studies of similar problems, that spacetime has exactly one timelike and an arbitrary number D of spacelike dimensions. The basic idea of our argument is to make use of the dynamical SO(4) invariance of motion in a $1/r$ potential. If D differs from three, the Coulomb potential of a point source falls off as $r^{(2-D)}$ and the dynamical symmetry is broken. This leads to additional contributions to the Lamb shift and the perihelion shift of planetary motion.

The standard experiments to test the $1/r$ potential are the Cavendish-Eötvös experiment for gravitation and the Cavendish experiments for electrostatics. Whereas the former leads only to the following bound for D [14]:

$$|D-3| \leqslant 10^{-4} \qquad \text{for } 6\text{ mm} < r_0 < 3\text{ cm} \tag{8}$$

(the meaning of r_0 will become clear in §§ 3 and 4), the latter is not able to detect any deviation of D from the value three. In fact this experiment tests only the validity of Gauss's law in the absence of charges [15]:

$$\int_S \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}a = 0. \tag{9}$$

Assuming gauge invariance, however, this law is fulfilled for any dimension D as both the definition of surface and volume integrals and the Coulomb potential V_{CB} change. To show this explicitly we use the generalised divergence theorem:

$$\int_S \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}a = \int_V \boldsymbol{\partial} \cdot \boldsymbol{E} \, \mathrm{d}^D v = \int_V \partial^i \partial^i V_{\mathrm{CB}} \, \mathrm{d}^D v \qquad i = 1, 2, 3. \tag{10}$$

Using the expression for the Coulomb potential derived in § 4 we thus obtain for any value of D

$$\int_S \boldsymbol{E} \cdot \boldsymbol{n} \, \mathrm{d}a = 2\pi^{D/2}(D-2)/\Gamma(D/2) \int_V \rho(r) \, \mathrm{d}^D v = 0 \tag{11}$$

if $\rho(r) = 0$. Thus the Cavendish experiment is only sensitive to the presence of mass terms or, e.g., self-interaction terms. As we obtain our results by dimensional continuation of the expressions for integer D this holds true also for non-integer values of D†.

3. The perihelion shift in $D = 3 + \varepsilon$ dimensions

We start with the perihelion shift in an arbitrary integer dimension D and use a rather obvious generalisation of the treatment given in a standard textbook [17] for three space dimensions.

† By measuring the rate of decrease of the apparent intensity of a light source, one obtains $|D-3| < 10^{-3}$ [16].

The problem we have to solve is that of geodesic motion outside of an $O(D)$-symmetric mass distribution. In the framework of the theory of gravity this solution is completely determined by the $(1+D)$-dimensional metric. Due to the $O(D)$ symmetry the most general ansatz for this metric is [18]

$$g_{\mu\nu} = \mathrm{diag}(e^{\nu}, -e^{\lambda}, -r^2, -r^2\sin^2\vartheta_1, -r^2\sin^2\vartheta_1\sin^2\vartheta_2, \ldots, \\ -r^2\sin^2\vartheta_1\sin^2\vartheta_2\ldots\sin^2\vartheta_{D-2}) \tag{12}$$

with the D-dimensional spherical coordinates $r, \vartheta_1, \vartheta_2, \ldots, \vartheta_{D-2}, \phi$. From equation (12) it is easy to calculate the Ricci curvature tensor. Only the components R_{00} and R_{11} are non-zero. Thus we only get two equations defining our D-dimensional Schwarzschild solution:

$$\begin{aligned} 0 &= R_{00} = [-\nu''/\nu' - (\nu'-\lambda')/2 - (D-1)/r]\nu' e^{\nu-\lambda}/2 \\ 0 &= R_{11} = \nu''/2 + \nu'/4 - \nu'\lambda'/4 - (D-1)/r. \end{aligned} \tag{13}$$

The solution to these equations is

$$\lambda = -\nu \qquad e^{\nu} = 1-(2m/r)(r_0/r)^{D-3} \tag{14}$$

r_0 being a constant length scale. Its value however can be roughly estimated as follows.

The size of, e.g., the mass of Mercury M_{mer} is derived from astronomical observations using the usual $1/r^2$ law. In these experiments length scales R typical for the Solar System are involved. From equation (14) we see that m_{mer}, M_{mer}, r_0 and R are related according to

$$M_{\mathrm{mer}} = m_{\mathrm{mer}}(r_0/R)^{D-3}. \tag{15}$$

We can therefore substitute

$$m_{\mathrm{mer}}(r_0/r)^{D-3} = M_{\mathrm{mer}}(R/r)^{D-3}. \tag{16}$$

In other words, if we use for m the usual masses we have to insert for r_0 the typical length scale at which these masses are measured. In the same way we conclude that, for the Lamb shift, r_0 has to be identified with the typical length scale involved in experiments measuring the electric charge.

Equation (14) determines the Schwarzschild metric for arbitrary integer $D \geq 2$. We now have to solve the problem of geodesic motion in this metric, e.g. we have to determine the solution of the variational equation

$$\delta \int \mathrm{d}s\,[e^{\nu}\dot{t}^2 - e^{-\nu}\dot{r}^2 - r^2(\dot{\vartheta}_1^2 + \dot{\vartheta}_2^2\sin^2\vartheta_1 + \ldots \dot{\phi}^2\sin^2\vartheta_1\ldots\sin^2\vartheta_{D-2})] = 0 \tag{17}$$

where a dot denotes a derivative with respect to the arc length s. The equations of motion resulting from this variation are

$$\begin{aligned} &\mathrm{d}/\mathrm{d}s\,(-r^2\sin^2\vartheta_1\sin^2\vartheta_2\ldots\sin^2\vartheta_{i-2}\dot{\vartheta}_i) = 2r^2\cos\vartheta_i\sin\vartheta_i \\ &\qquad \times(\sin^2\vartheta_1\sin^2\vartheta_2\ldots\sin^2\vartheta_{i-1}\dot{\vartheta}_{i+1}^2 + \ldots + \sin^2\vartheta_1\sin^2\vartheta_2\ldots \\ &\qquad \times\sin^2\vartheta_{i-1}\sin^2\vartheta_{i+1}\ldots\sin^2\vartheta_{D-2}\dot{\phi}^2) \qquad i=1,2,\ldots D-2 \\ &\mathrm{d}/\mathrm{d}s(e^{\nu}\dot{t}) = 0 \\ &\mathrm{d}/\mathrm{d}s(r^2\sin^2\vartheta_1\sin^2\vartheta_2\ldots\sin^2\vartheta_{D-2}\dot{\phi}) = 0. \end{aligned} \tag{18}$$

By an appropriate orientation we obtain

$$\vartheta_i = \pi/2 \qquad \dot{\vartheta}_i = 0 \qquad i=1,2,\ldots,D-2 \tag{19}$$

for some initial s. Then from equation (18) it follows that this holds for any s. Equation (19) is the D-dimensional definition of the plane of motion. With this choice we are left with only two equations valid in any integer dimension D:

$$\mathrm{d}/\mathrm{d}s(\mathrm{e}^{\nu}\dot{t})=0$$

$$\mathrm{d}/\mathrm{d}s(r^2\dot{\phi})=0 \tag{20}$$

with

$$\mathrm{e}^{\nu}=1-(2m/r)(r_0/r)^{D-3}.$$

The crucial point is that these equations can now be continued to non-integer dimensions by allowing D to be any real number. From now on we therefore substitute $D=3-\varepsilon$ and treat ε as a real parameter ($|\varepsilon|\ll 1$). Following the usual steps equations (20) are reduced to

$$r^2\dot{\phi}=h=\text{constant} \tag{21}$$

and

$$u''+u=(3-\varepsilon)mu^2(u_0/u)^{\varepsilon}+(m/h^2)(1-\varepsilon)(u_0/u)^{\varepsilon} \tag{22}$$

with

$$u=1/r \qquad u_0=1/r_0. \tag{23}$$

To solve equation (22) we make the usual ansatz for quasi-circular motion:

$$u(\phi)=A+B\cos(\omega\phi) \tag{24}$$

and expand equation (22) around $u=A$:

$$\begin{aligned}A+B(1-\omega^2)\cos(\omega\phi)&=(3-\varepsilon)A^2m(u_0/A)^{\varepsilon}+(1-\varepsilon)(m/h^2)(u_0/A)^{\varepsilon}\\&\quad+B\cos(\omega\phi)[(3-\varepsilon)(2-\varepsilon)mA(u_0/A)^{\varepsilon}-\varepsilon(1-\varepsilon)(m/h^2A)(u_0/A)^{\varepsilon}]\\&\quad+\mathrm{O}(B^2).\end{aligned} \tag{25}$$

For small values of ε, more precisely for

$$|\varepsilon\log(u_0/A)|\ll 1 \qquad \text{and} \qquad |\varepsilon|\ll 1 \tag{26}$$

we find

$$A=3A^2m+m/h^2+\mathrm{O}(\varepsilon)+\mathrm{O}(\varepsilon\log(u_0/A)) \tag{27}$$

or

$$A\simeq m/h^2+3m^3/h^4 \tag{28}$$

and we obtain for ω

$$\begin{aligned}1-\omega^2&=6mA-5mA\varepsilon-\varepsilon m/(Ah^2)+6mA\varepsilon\log(u_0/A)+\mathrm{O}(\varepsilon^2,\varepsilon^2\log(u_0/A))\\&\simeq 6(m/h)^2-5(m/h)^2\varepsilon-\varepsilon+6(m/h)^2\varepsilon\log(u_0/A).\end{aligned} \tag{29}$$

As $6(m/h)^2\simeq 10^{-7}$ the second term is negligible with respect to ε for reasonable values of u_0. We thus obtain

$$\omega^2=1-6(m/h)^2+\varepsilon$$

or

$$\omega=1-3(m/h)^2+\varepsilon/2. \tag{30}$$

The perihelion shift per revolution $\Delta\phi$ is thus determined by two contributions, one due to relativistic effects and one due to the fractional dimension of space:

$$\Delta\phi = 2\pi[3(m/h)^2 - \varepsilon/2] = \Delta\phi_0 - \pi\varepsilon \tag{31}$$

where $\Delta\phi_0$ is the usual relativistic perihelion shift. As the experimentally observed value for $\Delta\phi$ agrees with $\Delta\phi_0$ to within 0.5% for the planet Mercury [19], we finally get as a bound for ε

$$|\varepsilon| < 5\times 10^{-3}|\Delta\phi/\pi| \simeq 10^{-9}. \tag{32}$$

4. The Lamb shift in $D = 3 - \varepsilon$ dimensions

To analyse the consequences of a non-integer dimension for the Lamb shift in hydrogen it is necessary to include the spin of the electron, as otherwise one would not be able to distinguish between the generalised $2p_{1/2}$ and $2p_{3/2}$ states. It is also advisable to include relativistic effects. Although they are negligible in three space dimensions it is by no means clear that the changes due to a a non-vanishing ε are also small compared to the changes in the non-relativistic contributions. Thus to treat this problem we can, in principle, start with either a Schrödinger equation including relativistic and fine structure terms, or with the Dirac equation. As the spinor structure of the latter makes the generalisation to arbitrary dimensions difficult (although not impossible), we use the Schrödinger equation. In case the relativistic effects turn out to be important we would probably also have to analyse the fully relativistic Dirac equation.

We proceed as follows. First we generalise the standard treatment of the Lamb shift as found in reference [20] to arbitrary integer dimensionality of space $D = 3$, 4, 5, ... This step requires some group theory to find the higher-dimensional expressions for the fine structure term. We are then able to calculate the Lamb shift as a function of D: $\Delta E = \Delta E(D)$. This expression can be continued to arbitrary non-integer values of D and we evaluate

$$\Delta E(D = 3 - \varepsilon) - \Delta E(D = 3) \simeq -[\partial\Delta E/\partial D]_{D=3}\varepsilon. \tag{33}$$

Equation (33) together with the experimental bounds on additional contributions to the Lamb shift in hydrogen leads to an upper bound for $|\varepsilon|$.

We start with the three-dimensional Hamiltonian

$$(H_0 + W_1 + W_2 + W_3)\psi = E\psi \tag{34}$$

with

$$H_0 = -1/(2m)\nabla^2 + V \qquad V = -\alpha/r \tag{35}$$

$$W_1 = \nabla^2 V/(8m^2) \tag{36}$$

$$W_2 = -(E - V)^2/(2m) \tag{37}$$

$$W_3 = 1/(2m^2 r)\, \partial V/\partial r\, \boldsymbol{s}\cdot\boldsymbol{L}. \tag{38}$$

We start the generalisation to $D = 3, 4, 5, \ldots$, dimensions with the charge distribution $\rho(r)$ and the potential $V(r)$. $\rho(r)$ is normalised according to

$$\int \rho(r)\, \mathrm{d}^D r = e \tag{39}$$

and V is the solution of the D-dimensional Poisson equation

$$\partial^j\partial^j V=[2\pi^{D/2}/\Gamma(D/2)](D-2)r_0^{D-3}e\rho(r) \qquad j=1,2,3,\ldots,D. \tag{40}$$

In principle we should use an extended charge distribution like

$$\rho=[D\Gamma(D/2)/2\pi^{D/2}](e/R^D)\theta(R-r) \tag{41}$$

$$V=-\alpha r_0^{D-3}\begin{cases}1/r^{D-2} & \text{for } r\geqslant R\\ [DR^2-(D-2)r^2]/2R^D & \text{for } r\leqslant R.\end{cases} \tag{42}$$

For simplicity however we choose a point charge whenever this does not lead to infinities:

$$\rho(r)=e\delta^D(r) \qquad V=-(\alpha/r)(r_0/r)^{D-3}. \tag{43}$$

The generalisation of the angular momenta $s=\frac{1}{2}$, $L=1$, $j=\frac{1}{2}$ and $j=\frac{3}{2}$ to higher dimensions (named $(s=\frac{1}{2})_D$, $(L=1)_D$, $(j=\frac{1}{2})_D$ and $(j=\frac{3}{2})_D$) is most obvious if one uses Dynkin labels to denote the different representations of the group $O(D)$. One has to treat odd and even numbers of dimension separately as the corresponding rotation groups belong to different Lie algebras.

For odd space dimensions $D=2n+1$, $n=1,2,3,\ldots$, the corresponding algebra is B_n and the angular momentum states have to be identified with irreducible representations of the group as follows [21, 22]:

$$(L=1)_D\to(\underbrace{1\ 0\ \ldots\ 0\ 0}_{n}) \qquad \dim[(L=1)_D]=D$$
$$C_2[(L=1)_D]=D-1 \tag{44a}$$
$$(s=\tfrac{1}{2})_D\to(0\ 0\ \ldots\ 0\ 1) \qquad \dim[(s=\tfrac{1}{2})_D]=2^{(D-1)/2}$$
$$C_2[(s=\tfrac{1}{2})_D]=D(D-1)/8 \tag{44b}$$
$$(j=\tfrac{1}{2})_D\to(0\ 0\ \ldots\ 0\ 1) \qquad \dim[(j=\tfrac{1}{2})_D]=2^{(D-1)/2}$$
$$C_2[(j=\tfrac{1}{2})_D]=D(D-1)/8 \tag{44c}$$
$$(j=\tfrac{3}{2})_D\to(1\ 0\ \ldots\ 0\ 1) \qquad \dim[(j=\tfrac{3}{2})_D]=2^{(D-1)/2}(D-1)$$
$$C_2[(j=\tfrac{3}{2})_D]=D(D+7)/8. \tag{44d}$$

We have also listed the dimensions of the representations and the eigenvalues of the quadratic Casimir operator C_2.

For even space dimensions $D=2n$, $n=1,2,3,\ldots$, the algebra is D_n. In this case $(s=\frac{1}{2})_D$ and $(j=\frac{1}{2})_D$ have to be identified with different irreps:

$$(L=1)_D\to(\underbrace{1\ 0\ \ldots\ 0\ 0}_{n}) \qquad \dim[(L=1)_D]=D \qquad C_2[(L=1)_D]=D-1 \tag{45a}$$
$$(s=\tfrac{1}{2})_D\to(0\ 0\ \ldots\ 0\ 1) \qquad \dim[(s=\tfrac{1}{2})_D]=2^{(D/2-1)}$$
$$C_2[(s=\tfrac{1}{2})_D]=D(D-1)/8 \tag{45b}$$
$$(j=\tfrac{1}{2})_D\to(0\ 0\ \ldots\ 1\ 0) \qquad \dim[(j=\tfrac{1}{2})_D]=2^{(D/2-1)}$$
$$C_2[(j=\tfrac{1}{2})_D]=D(D-1)/8 \tag{45c}$$
$$(j=\tfrac{3}{2})_D\to(1\ 0\ \ldots\ 0\ 1) \qquad \dim[(j=\tfrac{3}{2})_D]=2^{(D/2-1)}(D-1)$$
$$C_2[(j=\tfrac{3}{2})_D]=D(D+7)/8. \tag{45d}$$

Inserting these values into the generalisation of $2\boldsymbol{s}\cdot\boldsymbol{L}$, namely

$$(2\boldsymbol{s}\cdot\boldsymbol{L})_D = C_2[(j)_D] - C_2[(s)_D] - C_2[(L)_D] \tag{46}$$

we obtain

$$(2\boldsymbol{s}\cdot\boldsymbol{L})_D = \begin{cases} 0 & \text{for } (2\mathrm{s}_{1/2})_D \\ D-1 & \text{for } (2\mathrm{p}_{1/2})_D \\ 1 & \text{for } (2\mathrm{p}_{3/2})_D. \end{cases} \tag{47}$$

Using equation (47) we are able to generalise the Hamiltonian (34) to

$$(H_{0,D} + W_{1,D} + W_{2,D} + W_{3,D})\psi = E\psi \tag{48}$$

with

$$H_{0,D} = -1/(2m)(\partial_r^2 + (D-1)/r\partial_r) + C_2[(L)_D]/(2mr^2) + V \tag{49}$$

$$W_{1,D} = e/(8m^2)\, 2\pi^{D/2}/[\Gamma(D/2)](D-2)r_0^{D-3}\rho(r) \tag{50}$$

$$W_{2,D} = -(E-V)^2/(2m) \tag{51}$$

$$W_{3,D} = \begin{cases} 0 & \text{for } (2\mathrm{s}_{1/2})_D \\ (D-1)/(4m^2r)\,\partial V/\partial r & \text{for } (2\mathrm{p}_{1/2})_D. \end{cases} \tag{52}$$

Let us now turn to the eigenvalues of the energy which can be expressed as expectation values of the Hamiltonian (48):

$$E_{i,D} = \int \mathrm{d}^D r\, \psi_{i,D}^* H_D \psi_{i,D} \tag{53}$$

with the normalisation condition

$$\int \mathrm{d}^D r\, \psi_{i,D}^* \psi_{i,D} = 1. \tag{54}$$

Due to the rotational symmetry, all angular integrals in (53) and (54) can be carried out, leaving only radial integrals to be performed:

$$E_{i,D} = r_0^{3-D} \int \mathrm{d}r\, r^{D-1} f_{i,D}(r) H_{i,D}(r) f_{i,D}(r) \tag{55}$$

with

$$r_0^{3-D} \int \mathrm{d}^D r\, r^{D-1}(f_{i,D})^2 = 1 \qquad D = 3, 4, 5 \ldots. \tag{56}$$

The crucial point is that these integrals can be continued to arbitrary dimensions. We indicate this continuation by replacing $E_{i,D}$ by $E_i(D)$, etc:

$$E_i(D) = r_0^{3-D} \int \mathrm{d}r\, r^{D-1} f_i(D, r) H_i(D, r) f_i(D, r) \tag{57}$$

with

$$r_0^{3-D} \int \mathrm{d}^D r\, r^{D-1}(f_i(D, r))^2 = 1 \qquad D = \text{real number}. \tag{58}$$

As D is very close to three it is sufficient to expand equations (57) and (58) around this value. With

$$D=3-\varepsilon \tag{59}$$

we have

$$\begin{aligned}E_i(3-\varepsilon) &\simeq E_i(3)-\partial E_i/\partial D|_{D=3}\varepsilon\\ &= E_i(3)-\varepsilon\bigg(\int \mathrm{d}r\, r^2 \ln(r/r_0) E_i(f_i(3,r))^2\\ &\quad +2\int \mathrm{d}r\, r^2\, \partial f_i(D,r)/\partial D|_{D=3} E_i f_i(3,r)\\ &\quad +\int \mathrm{d}r\, r^2 f_i(3,r)\partial H_i(D,r)/\partial D|_{D=3} f_i(3,r)\bigg)\end{aligned} \tag{60}$$

and

$$\int \mathrm{d}r\, r^2 \ln(r/r_0)(f_i(3,r))^2+2\int \mathrm{d}r\, r^2 \partial f_i(D,r)/\partial D|_{D=3} f_i(3,r)=0. \tag{61}$$

Inserting (61) into (60) reduces the latter to

$$E_i(3-\varepsilon)-E_i(3)=-\varepsilon\int \mathrm{d}r r^2 f_i(3,r)\partial H_i(D,r)/\partial D|_{D=3} f_i(3,r) \tag{62}$$

which shows that the Hellmann–Feynman theorem is valid for our problem. Using this equation we can now calculate the contributions from $H_{0,D}$, $W_{1,D}$, $W_{2,D}$ and $W_{3,D}$ to the Lamb shift:

$$\begin{aligned}\Delta E(H_{0,D}) &= -\varepsilon\int \mathrm{d}r\, r^2 f(2\mathrm{p}_{1/2})\\ &\quad \times[-1/(2mr)\partial_r+1/(2mr^2)-(\alpha/r)\log(r_0/r)]f(2\mathrm{p}_{1/2})\\ &\quad +\varepsilon\int \mathrm{d}r\, r^2 f(2\mathrm{s}_{1/2})[-1/(2mr)\partial_r-(\alpha/r)\log(r_0/r)]f(2\mathrm{s}_{1/2})\end{aligned} \tag{63}$$

$$\begin{aligned}\Delta E(W_{1,D}) &= -(\varepsilon\alpha/8m^2R^3)\int \mathrm{d}r\, r^2 f(2\mathrm{p}_{1/2})[4+3\log(r_0/R)]f(2\mathrm{p}_{1/2})\\ &\quad +(\varepsilon\alpha/8m^2R^3)\int \mathrm{d}r\, r^2 f(2\mathrm{s}_{1/2})[4+3\log(r_0/R)]f(2\mathrm{s}_{1/2})\end{aligned} \tag{64}$$

$$\begin{aligned}\Delta E(W_{2,D}) &= -\varepsilon\int \mathrm{d}r\, r^2 f(2\mathrm{p}_{1/2})[(E(\mathrm{p}_{1/2})+\alpha/r)/m][\partial E(2\mathrm{p}_{1/2})/\partial D|_{D=3}\\ &\quad +(\alpha/r)\log(r_0/r)]f(2\mathrm{p}_{1/2})\\ &\quad +\varepsilon\int \mathrm{d}r\, r^2 f(2\mathrm{s}_{1/2})[(E(2\mathrm{s}_{1/2})+\alpha/r)/m][\partial E(2\mathrm{s}_{1/2})/\partial D|_{D=3}\\ &\quad +(\alpha/r)\log(r_0/r)]f(2\mathrm{s}_{1/2})\end{aligned} \tag{65}$$

$$\begin{aligned}\Delta E(W_{3,D}) &= -\varepsilon\int \mathrm{d}r\, r^2 f(2\mathrm{p}_{1/2})[-1/(4m^2r)](\alpha/r^2)\\ &\quad \times[3+2\log(r_0/r)]f(2\mathrm{p}_{1/2}).\end{aligned} \tag{66}$$

We have approximated the physical extended charge distribution by a point charge in equations (63), (65) and (66)

Inserting the radial functions

$$f(2p_{1/2}) = (1/2a^3)^{1/2}[1-(r/2a)]\,e^{-(r/2a)} \tag{67}$$

and

$$f(2s_{1/2}) = (1/24a^5)^{1/2} r\,e^{-(r/2a)} \tag{68}$$

with

$$a = 1/m\alpha = 5.292 \times 10^{-9}\ \mathrm{cm} \tag{69}$$

we obtain

$$\Delta E(H_{0,D}) = -\varepsilon\alpha/12a \tag{70}$$

$$\Delta E(W_{1,D}) = \Delta E(H_{0,D})(\alpha^2/8)[4+3\log(r_0/R)]\,e^{-R/a}[(R/a)-1] \tag{71}$$

$$\begin{aligned}\Delta E(W_{2,D}) &= \Delta E(H_{0,D})\alpha^2[2\psi(1)-2\log(r_0/a)-\tfrac{23}{8}]\\ &\quad+[\Delta E(H_{0,D})+\Delta E(W_{1,D})+\Delta E(W_{2,D})+\Delta E(W_{3,D})]3\alpha^2/8\end{aligned} \tag{72}$$

$$\Delta E(W_{3,D}) = \Delta E(H_{0,D})(\alpha^2/8)[2\psi(2)-2\log(r_0/a)-3]. \tag{73}$$

For $r_0 > 10^{-37}$ fm the contributions (71)-(73) are less than one per cent of $\Delta E(H_{0,D})$. (For $r_0 > 10^{-40}$ fm they are less than ten per cent.) As we discussed earlier that r_0 has to be much larger than these values, we can safely neglect the contributions from W_1, W_2 and W_3 to the Lamb shift. Our result thus becomes

$$\Delta E = -2.27\varepsilon\ \mathrm{eV}. \tag{74}$$

As the theoretical and experimental values for the Lamb shift agree to within [23, 24]

$$|\Delta E(\exp) - \Delta E(\mathrm{theor})| < 0.02\ \mathrm{MHz} = 8.2\times 10^{-11}\ \mathrm{eV} \tag{75}$$

we get a rather stringent bound for the deviation of the space dimension from the value three:

$$|\varepsilon| < 3.6\times 10^{-11}. \tag{76}$$

We emphasise that this result relies heavily on the experimental verification of Gauss's law, implying that the Coulomb potential differs from the $1/v$ law in $D \neq 3$ dimensions. This contradicts the assumptions made by Herrick [25] and Stillinger [26] who found no effect of $D \neq 3$ in the spectrum of the hydrogen atom.

5. Conclusion

We have shown that the dynamical symmetry associated with motion in a $1/r$ potential provides extremely stringent limits for any possible deviation of the number of space dimensions D from the integer value three. For length scales of the order of the distance Mercury-Sun (6×10^7 km) we obtained $|\varepsilon| = |D-3| < 10^{-9}$ and for those of the order of the Bohr radius 5×10^{-11} m we obtained $|\varepsilon| < 4\times 10^{-11}$. As ε should vary very slowly as a function of the typical length scales, as long as these scales are large compared to 10^{-30} m our second bound rules out the interpretation proposed in reference [12]. There the authors assumed ε to be of the order of 5×10^{-7} on a length scale characterised by the Compton wavelength of the electron, namely 4×10^{-13} m. Apart from this result, we believe that our analysis is rather important for the microscopic models of spacetime mentioned in § 1. However, much work still has to be done. It should be possible to derive for each of these models the function $\varepsilon(l_0)$, where l_0 is the typical length scale of a physical problem, and thus to decide whether these models are compatible with our bounds. Moreover it is important to find upper limits for $|\varepsilon|$ at much smaller length scales (e.g. from the anomalous magnetic moment of the muon one obtains $|\varepsilon|<10^{-5}$ for $l_0 \simeq 2\times 10^{-15}$ m).

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